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A tribute to Géza Freud

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Abstract

We discuss some of the recent work in approximation theory motivated by the research of Géza Freud (1922–1979).
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1. Introduction

Géza Freud (1922–1979) was one of the main leaders in approximation theory of the past century. His interests included trigonometric and polynomial approximation, summability theory, harmonic analysis, differential equations, functional analysis, in short, almost everything connected with approximation theory at that time. Most of the articles in Volumes 46 and 48 of Journal of Approximation Theory deal with his contributions to this field.

During the last 10 years or so of his life, he was "obsessed" (in his words) with the theory of weighted polynomial approximation. This subject itself has diversified a lot since his times, with the infusion of potential theory [58,59,79] and Hilbert transform techniques (cf. [17] and references therein). Several books have been published dealing with various aspects of weighted polynomial approximation. The book [50] gives a general, more or less historical, introduction to the different ideas in the development of this theory. It also contains a fairly large bibliography of 356 items. The treatise [66] of Saff and Totik deals with the potential theory aspects, while the treatise [33] of Levin and Lubinsky gives an extensive treatment of orthogonal

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polynomials. In addition, there are many surveys dealing with more specialized topics, for example, [11,15,17,36,37,46,71].

In this paper, we will discuss some current research that appears to be motivated by the problems and ideas which Freud worked on. We will only focus on the classical approximation problems which Freud himself had contributed to, but not attempt to write an exhaustive review. In particular, the impressive results by Benko [2] on the closure of functions of the form $w^n P_n$, P_n being a polynomial of degree n, and Totik [78] on the Christoffel functions with respect to variable weights will not be discussed. In light of the different books and surveys published after the special issues of this Journal in 1986, we will further focus on the research after 1996, when [50] was published.

Some of the new features of the current research in weighted approximation appear to be the presence of endpoint effects in the degree of approximation with respect to weights with a fast decay near infinity, multivariate approximation, orthogonal polynomials, interpolation, and another process which I will call quasi-interpolation. In this paper, I will give examples of the kind of results proved recently in these directions.

2. Weighted polynomial approximation

The Bernstein approximation problem seeks necessary and sufficient conditions on a weight function w to ensure that the class of weighted polynomials $\{wP$: P a polynomial$\}$ is dense in the space $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$, vanishing at infinity. The solution by Riesz [65] is most complete in the case when one considers $L^2(\mathbb{R})$ rather than $C_0(\mathbb{R})$, and is given in terms of the "Christoffel function", defined for integer $n \geqslant 1$ and $z \in \mathbb{C}$ by

$$\lambda_n(w, z) := \min |P(z)|^{-2} \int |P(t)|^2 w(t)\, dt, \tag{2.1}$$

where the minimum is taken over all polynomials of degree $< n$. The Christoffel function can be evaluated explicitly using orthonormal polynomials with respect to w (cf. [24, Theorem I.4.1]). Under some very general conditions on the weight function, the weight function can be determined from the sequence of Christoffel functions (cf. [24,33,63,78]). For many other applications of the Christoffel function, we refer to [64]. Freud initiated an ambitious program of estimating the degree of approximation by weighted polynomials where the polynomials are of a given degree. His hope was to be able to formulate the conditions as far as possible in terms of the Christoffel functions, without which no estimate on the degree of approximation can be obtained. A compromise was to formulate the conditions more directly in terms of the weight function itself. The class of weight functions was expanded gradually, starting with the Hermite weight. Freud's papers clearly show a lot of effort in coming up with the "right" class of weight functions. A large class of weight functions arising out of this effort is now called the class of Freud weights (Definition 2.1).

In the sequel, we adopt the following convention regarding constants. The symbols $c, c_1, \ldots$ will denote positive constants depending only on the fixed quantities in question, such as the weight function, the norms, and smoothness parameters. Their value may be different at different occurrences, even within the same formula. The notation $A \sim B$ will mean $c_1 A \leqslant B \leqslant c_2 A$.

Definition 2.1. Let $w : \mathbb{R} \to (0, \infty)$. We say that w is a *Freud weight* if $Q(x) := \log(1/w(x))$ is an even, convex function on $\mathbb{R}$, Q is twice continuously differentiable on $(0, \infty)$ and there are positive constants c, c_1 such that

$$1 \leqslant c \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant c_1 \tag{2.2}$$

for all $x > 0$.

The prototypical examples of a Freud weight are $\exp(-|x|^\alpha)$, $\alpha > 1$. Sometimes, one requires the differentiability conditions and (2.2) only for $x \geqslant c_2$. However, it is proved in [50, Proposition 3.1.3] that if w is such a weight, there exists a Freud weight $\bar{w}$ such that $w(x) \sim \bar{w}(x)$ for all $x \in \mathbb{R}$. An interesting feature of the theory of the degree of weighted polynomial approximation with Freud weights is that the end results in this theory parallel closely the corresponding theory of trigonometric polynomial approximation. In particular, unlike the theory of polynomial approximation on $[-1, 1]$, there is no "end-point effect" in this theory.

The class of weights is now far more generalized, mostly in the direction of examining the case when $|x|^\alpha$ is replaced by another function that tends to infinity faster than a power of x as $x \to \infty$. A good survey of this area can be found in the dissertation of Mashele [46]. To give the reader a glimpse of the recent results, we describe a direct theorem, due to Damelin and Lubinsky [16], for "Erdős weights". It turns out that unlike Freud weights, approximation by Erdős weights show an "end point effect", even though there is no reason, obvious from the definitions, as to why this should be the case.

Definition 2.2. A function $W := e^{-Q}$ is called an *Erdős weight* if each of the following conditions is satisfied.

(a) $Q : \mathbb{R} \to \mathbb{R}$ is an even and differentiable function, and $Q'(x) > 0$ if $x \in (0, \infty)$.
(b) The function $xQ'(x)$ is strictly increasing in $(0, \infty)$, and $\lim_{x \to 0+} xQ'(x) = 0$.
(c) Let $T(x) := T_Q(x) := xQ'(x)/Q(x)$. There exist constants $c, c_1 > 0$ such that for every $c < x < y$, $T(x) \leqslant c_1 T(y)$. Also, $T(x) \to \infty$ as $x \to \infty$.
(d) There exist constants $c_2, c_3 > 0$ such that for $c \leqslant x \leqslant y$,

$$\frac{yQ'(y)}{xQ'(x)} \leqslant c_3 \left(\frac{Q(y)}{Q(x)} \right)^{c_2}.$$

Prototypical Erdős weights are $W(x) = \exp(-\exp_k(|x|^\alpha))$, where $\exp_k$ denotes the k times iterated exponential function, $k \geqslant 1$, and $\alpha > 0$. Another example is $W(x) = \exp(-\exp((\log(1+x^2))^\beta)$, $\beta > 1$. Erdős weights decay faster near infinity than the Freud weights. Moreover, it is interesting to note that they need not be twice differentiable on $(0, \infty)$.

For any weight $W = \exp(-Q)$ such that $xQ'(x) \uparrow \infty$ as $x \uparrow \infty$, the numbers a_u are defined to be the least positive solution of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\frac{dt}{\sqrt{1-t^2}}, \quad u > 0, \tag{2.3}$$

and the inverse of the function $u \mapsto a_u/u$ is defined by $\sigma(t) := \inf\{u\colon a_u \leqslant ut\}$. To make the end point effect more precise, we now define a modulus of smoothness.

For $p > 0$, and a measurable function f on a measurable subset A of $\mathbb{R}$, we write

$$\|f\|_{p,A} := \begin{cases} \left\{\int_A |f(t)|^p dt\right\}^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname*{ess\,sup}_{t \in A} |f(t)| & \text{if } p = \infty. \end{cases}$$

For $h > 0$, $f : \mathbb{R} \to \mathbb{R}$, and integer r, we define the rth forward difference of f by

$$\Delta_h^r(f, x) := \sum_{\ell=0}^{r} \binom{r}{j} (-1)^j f(x - \ell h + rh/2).$$

The class of polynomials of degree at most n will be denoted by Π_n. The end point effect is encoded in the modulus of smoothness by letting the step size h depend upon x, analogous to the case of the Ditzian–Totik modulus of smoothness for approximation on $[-1, 1]$. Accordingly, we write

$$\Phi_t(x) := \left(1 - \frac{|x|}{\sigma(t)}\right)^{1/2} + T(\sigma(t))^{-1/2}, \tag{2.4}$$

and define the modulus of smoothness of order r of a function $f \in L^p(\mathbb{R})$ by

$$\omega_{r,p}(f, W, t) := \sup_{0<h<t} \|W\Delta^r_{h\Phi_t(x)}(f, x)\|_{p,|x| \leqslant \sigma(2t)} + \inf_{P \in \Pi_{r-1}} \|(f-P)W\|_{p,|x| \geqslant \sigma(4t)}. \tag{2.5}$$

The direct theorem can be stated as follows.

Theorem 2.1. *Let W be an Erdős weight, $r \geqslant 1$, $0 < p < \infty$, $f : \mathbb{R} \to \mathbb{R}$, $fW \in L^p(\mathbb{R})$. Then for every integer $n \geqslant r-1$, there exists a polynomial $P \in \Pi_n$ such that*

$$\|(f-P)W\|_{p,\mathbb{R}} \leqslant c_1 \omega_{r,p}\left(f, W, c\frac{a_n}{n}\right). \tag{2.6}$$

The same estimate holds also for $p = \infty$ if we assume that $fW \in C_0(\mathbb{R})$.

The proof of this theorem is quite technical. As with the case of the Freud weights, it involves the approximation of W^{-1}, which is done by interpolation of the corresponding Lubinsky entire function (cf. [50, Section 8.2]). The proof then relies upon a classical technique as in [20], where results on the approximation of the characteristic function of an interval are combined with those on the approximation of f by piecewise constants.

The "converse theorem" corresponding to Theorem 2.1 was proved by Damelin [8] under some additional conditions on the weight functions. Damelin has kindly informed me that he and his collaborators have made further generalizations and applications of this theory [5,12]. Mastroianni and Szbados [47,48] have extended other such ideas as one-sided approximation, used by Freud in his work on the degree of approximation.

3. Orthogonal polynomials

Freud's theory of weighted polynomial approximation was inseparable from his theory of orthogonal polynomials with respect to the weights involved. A very detailed survey about Freud's orthogonal polynomials and their generalizations can be found in [33,37]. A hierarchy of weight functions on both finite and infinite intervals is studied in a unified manner in [33]. A survey from the point of view of a proof technique based on the Riemann–Hilbert problem can be found in [17]. In the beginning, a driving factor in this research was the *Freud conjecture* regarding the asymptotic behavior of the recurrence coefficients of polynomials orthonormal with respect to $\exp(-|x|^\beta)$ for $\beta > 0$. This conjecture itself was settled in [44]. Further research [45,50,77] focused on the study of the asymptotic behavior of the leading coefficients of the orthonormalized polynomials. The following Theorem 3.1 by Kriecherbauer and McLaughlin [29] gives a relatively recent result in this direction. The connection between the behavior of the leading coefficients and the zeros of orthogonal polynomials is described by Stahl and Totik [67], and Andrievskii and Blatt [1]. Further, it is obvious that a detailed information regarding orthogonal polynomials is vital for the investigation of such processes of approximation as orthogonal expansions and interpolation.

Theorem 3.1. *Let $\beta > 0$ and $\{p_n(x) \coloneqq \gamma_n x^n + \cdots \in \Pi_n\}$ be the system of polynomials orthonormal on $\mathbb{R}$ with respect to $\exp(-k_\beta |x|^\beta)$, where $k_\beta \coloneqq \frac{\Gamma(\beta/2)\sqrt{\pi}}{\Gamma((\beta+1)/2)}$. Let*

$$C_\beta \coloneqq \frac{2}{\pi}(-1)^{n+1}\Gamma(\beta+1)\left(\frac{\beta-1}{2\beta}\right)^{\beta+1}\left(\sum_{j=0}^{\infty}(2j+1-\beta)^{-1}\prod_{\ell=1}^{j}\frac{2\ell-1}{2\ell}\right)\cos(\pi\beta/2).$$

(a) *We have*

$$\gamma_n\sqrt{\pi}n^{(2n+1)/(2\beta)}e^{-n/\beta}2^{-n} = 1 + \left(\frac{\beta-4}{24\beta}\right)\frac{1}{n} + err_{\beta,n}, \tag{3.1}$$

where

$$err_{\beta,n} = \begin{cases} \mathcal{O}(n^{-2}) & \text{if } 0<\beta\leqslant 1/2, \\ \mathcal{O}(n^{-1/\beta}) & \text{if } 1/2<\beta<1, \\ \dfrac{(-1)^{n+1}}{4n(\log n)^2}(1+o(1)) & \text{if } \beta=1, \\ C_\beta n^{-\beta}+\mathcal{O}(n^{1-2\beta}) & \text{if } 1<\beta\leqslant 3/2, \\ C_\beta n^{-\beta}+\mathcal{O}(n^{-2}) & \text{if } 3/2<\beta\leqslant 2, \\ \mathcal{O}(n^{-2}) & \text{if } \beta\geqslant 2. \end{cases} \tag{3.2}$$

(b) *For* $\beta\geqslant 1$, *the zeros* $x_{1,n}>x_{2,n}>\cdots>x_{n,n}$ *of* p_n *satisfy for each* $k=1,\dots,n$,

$$\frac{x_{k,n}}{n^{1/\beta}} = 1-(2\beta^2)^{-1/3}\frac{\imath_k}{n^{2/3}}+\mathcal{O}(n^{-1}), \tag{3.3}$$

where $-\imath_k$ is *the kth zero of the Airy function Ai.*

Analogous results for more general weights are obtained by Damelin in [10].

4. Interpolation

The theory of Lagrange interpolation at the zeros of Hermite polynomials was studied by Freud [23] and Nevai [62,64]. The theory remained largely dormant for many years, except for an occasional paper [28,61]. Apparently, the theory has received a lot of recent attention by Damelin, Lubinsky, Mastroianni, Szabados, Vértesi, among others. For a relatively recent survey of this area, we refer to the paper [71] of Szabados.

An important characteristic of interpolation is the Lebesgue function and its norm, the Lebesgue constant. For $n=1,2,\dots$, let $Y_n := \{y_{j,n}\colon j=1,\dots,n\}$ be a set of distinct real numbers. Then for every integer $n\geqslant 1$, there exist polynomials $\ell_{j,n}(Y_n)$ of degree at most $n-1$ such that $\ell_{j,n}(Y_n;y_{k,n})=0$ if $k\neq j$ and $\ell_{j,n}(Y_n;y_{j,n})=1$. The Lebesgue function of this system with a weight W is defined by

$$\Lambda(Y_n,W,x) := W(x)\sum_{j=1}^{n} W^{-1}(y_{j,n})|\ell_{j,n}(Y_n;x)|, \tag{4.1}$$

and the corresponding Lebesgue constant is defined by $\Lambda(Y_n,W) := \|\Lambda(Y_n,W,\cdot)\|_{\infty,\mathbb{R}}$. It is easy to see that for any f with $Wf\in C_0(\mathbb{R})$,

$$\left|W(x)\left(f(x)-\sum_{j=1}^{n} f(y_{j,n})\ell_{j,n}(x)\right)\right| \leqslant (1+\Lambda(Y_n,W,x))\min_{P\in\Pi_n}\|(f-P)W\|_{\infty,\mathbb{R}}.$$

Thus, the Lebesgue function determines for what functions and at what points the Lagrange interpolation polynomials converge.

Let $\{p_n(x)=\gamma_n x^n+\cdots+\in\Pi_n\}$ be the system of polynomials orthonormal on $\mathbb{R}$ with respect to the weight function W^2, $\mathcal{X}_n$ be the set of zeros, $x_{1,n}>\cdots>x_{n,n}$, of p_n,

$n = 1, 2, \ldots$. In the case when W is a Freud weight, Matijla [49] proved that $\Lambda(\chi_n, W) = \mathcal{O}(n^{1/6})$. Szabados [70] proved this order of magnitude to be exact; i.e., $\Lambda(\chi_n, W) \sim n^{1/6}$. One of the most remarkable result of [70] is that the Lebesgue constant for the system of zeros of orthogonal polynomials together with certain "end points" is of the order $\log n$.

Theorem 4.1. *Let W be a Freud weight, x_0 be a positive number such that*

$$|p_n(x_0)|W(x_0) = ||p_n W||_{\infty,\mathbb{R}},$$

and $\mathcal{V}_{n+2} := \chi_n \cup \{x_0, -x_0\}$. Then

$$\Lambda(\mathcal{V}_{n+2}, W) \sim \log n. \tag{4.2}$$

In the case when W is the Hermite weight, $W(x) = \exp(-x^2/2)$, Szabados [70] proved further that the norm of any Π_n valued projection operator with respect to the weighted supremum norm is at least $c \log n$. Vértesi [81] has proved a pointwise lower bound on the Lebesgue function in the general case, analogous to the corresponding results for the compact interval (cf. [73, Theorem 3.2, p. 72]).

Theorem 4.2. *Let W be a Freud weight, the numbers a_u be defined as in* (2.3) *and* $\varepsilon > 0$. *Then for any set of interpolation points, Y_n, there exist sets $H_n := H_n(W, \varepsilon, Y_n)$ with $|H_n| \leqslant \varepsilon a_n$ such that*

$$\Lambda(Y_n, W, x) > \frac{\varepsilon}{3840} \log n, \quad x \in [-a_n, a_n] \backslash H_n, \ n \geqslant c. \tag{4.3}$$

In the case of Erdős weights with some additional conditions, the analogues of Theorems 4.1 and 4.2 were proved by Damelin [7] and Vértesi [80], respectively. Horváth and Szabados [26] have proved an analogue of Theorem 4.1 for another class of general weights. The order of magnitude of the Lebesgue function for the zeros of orthogonal polynomials has been evaluated in a great generality by Kubayi [30,31]. The question of mean convergence of Lagrange interpolation processes is recently studied by Kubayi and Lubinsky [32], Damelin et al. [15], Lubinsky [39], and Lubinsky and Mastroianni [42,43] among others. The analogues of classical results regarding other kinds of interpolation, for example, Hermite or Hermite Fejér interpolation, are also being studied for weights on the real line, for example, we are aware of [13,14,27,70]. Some applications to the convergence of quadrature formulas and other related topics are discussed in [35,38].

5. Quasi-interpolation

In [22], Freud studied a sequence of linear operators L_n on $C[-1, 1]$ corresponding to interpolation nodes Y_n with the following properties: (a) For $P \in \Pi_{cn}$, $L_n(P) = P$, (b) for each $f \in C[-1, 1]$, $L_n(f) \in \Pi_{c_1 n}$, (c) $L_n(f; y_{j,n}) = f(y_{j,n})$ for $j = 1, \ldots, n$, and (d)

$\|L_n(f)\|_{\infty,[-1,1]} \leqslant c_2 \|f\|_{\infty,[-1,1]}$. This theme was carried out by many mathematicians ([73, Chapter II]). For example, the following theorem of Erdős et al. [19] gives necessary and sufficient conditions for the existence of a convergent sequence of interpolatory polynomials.

Theorem 5.1. *Let* $x_{k,n} = \cos\theta_{k,n}$ *be distinct points on* $[-1,1]$, $x_{k,n} > x_{k+1,n}$, $k = 1,\ldots,n-1$, $n = 1,2,\ldots$ *. The following are equivalent.*

(a) *To every* $f \in C[-1,1]$ *and* $\varepsilon > 0$, *there exists a sequence of polynomials* $r_n \in \Pi_{n(1+\varepsilon)}$ *such that* $r_n(x_{k,n}) = f(x_{k,n})$ *for* $k = 1,\ldots,n$, *and* $\|f - r_n\|_{\infty,[-1,1]} \to 0$ *as* $n \to \infty$.

(b) *We have*

$$\limsup_{n\to\infty} \frac{\#\{k:\ \theta_{k,n} \in I_n\}}{n|I_n|} \leqslant \frac{1}{\pi} \tag{5.1}$$

for every sequence of intervals $I_n \subseteq [0,\pi]$ *for which* $\lim_{n\to\infty} n|I_n| \to \infty$, *and*

$$\liminf_{n\to\infty}\ n(\theta_{k,n+1} - \theta_{k,n}) > 0, \quad 1 \leqslant k \leqslant n. \tag{5.2}$$

In the implication (b)⇒(a), a lot more can be said than convergence. The following Theorem 5.2 is a consequence of Theorem 5.1 and a result of Szabados [69] (cf. [73, Theorem 2.7, p. 52]).

Theorem 5.2. *Let* $x_{k,n} = \cos\theta_{k,n} \in [-1,1]$ *be an arbitrary system of nodes* $(0 \leqslant \theta_{1,n} < \cdots < \theta_{n,n} \leqslant \pi)$ *and let*

$$d_n := \min_{1\leqslant k\leqslant n-1} \theta_{k+1,n} - \theta_{k,n}.$$

Then for any $\varepsilon > 0$, *there exist linear polynomial operators* P_n *on* $C[-1,1]$ *with the following properties*: (a) *If* $m = \lfloor \pi(1+\varepsilon)/d_n \rfloor$ *then* $P_n(P) = P$ *for all* $P \in \Pi_m$, (b) *for* $f \in C[-1,1]$, $P_n(f) \in \Pi_N$ *where* $N = (\pi/d_n + 1)(1+3\varepsilon)$, (c) $P(f, x_{k,n}) = f(x_{k,n})$ *for* $k = 1,\ldots,n$, *and* (d) $\|P_n(f)\|_{\infty,[-1,1]} \leqslant c\|f\|_{\infty,[-1,1]}$.

In the case when $x_{k,n}$'s are the zeros of Chebyshev polynomials, an explicit construction for an analogous operator was given by Szabados [68]. Further results in the context of weighted polynomial approximation can be found in [74–76,82].

We will discuss two more recent deviations on this theme.

In [55], we proved that an analogue of Theorem 5.2 holds in practically any setting where the Jackson theorem holds, provided we drop the requirement of linearity. Let $d \geqslant 1$ be an integer, $K \subset \mathbb{R}^d$ be compact, $r \geqslant 1$ be an integer, and let $\mathscr{D}$ be a linear (partial) differential operator of order r defined for a subset of $C(K)$, and having coefficient functions in $C(K)$. We say that a sequence $\{V_n\}$ of subspaces of $C(K)$ has the *Jackson property* with respect to $\mathscr{D}$ if for every f in the domain of $\mathscr{D}$, and $n \geqslant 1$, one has

$$\inf_{v\in V_n} \|f - v\|_{C(K)} \leqslant An^{-r}\{\|f\|_{C(K)} + \|\mathscr{D}(f)\|_{C(K)}\},$$

where A is a positive constant, independent of n and f. Our theorem can now be stated as follows.

Theorem 5.3. *Suppose that $\mu_1, \ldots, \mu_N$ are compactly supported Borel measures on $K \subseteq \mathbb{R}^d$, and that $S_j := \mathrm{supp}(\mu_j)$, $j = 1, \ldots, N$, are mutually disjoint. Assume that*

$$\frac{\beta}{n} \leqslant \eta := \min_{1 \leqslant i,j \leqslant N} \mathrm{dist}(S_i, S_j) \leqslant B, \tag{5.3}$$

for some positive integer n and positive constants β and B (which may depend on K and d, but not on N, μ_j's or S_j's). Let $r \geqslant 1$ be an integer, and let $\mathscr{D}$ be a linear (partial) differential operator of order r defined for a subset of $C(K)$, and having coefficient functions in $C(K)$. Further, let $\{V_k\}$ be a sequence of finite-dimensional subspaces of $C(K)$ having the Jackson property with respect to $\mathscr{D}$. If $\alpha > 0$, there exists a positive constant $C := C(\alpha, \beta, d, B, \mathscr{D})$ with the following property: For every $f \in C(K)$, there exists $\mathsf{P}_n(f) \in V_{Cn}$ such that

$$\int f d\mu_j = \int \mathsf{P}_n(f)\, d\mu_j, \quad j = 1, \ldots, N, \tag{5.4}$$

and

$$\|f - \mathsf{P}_n(f)\|_{C(K)} \leqslant (2 + \alpha) \inf_{v \in V_{Cn}} \|f - v\|_{C(K)}. \tag{5.5}$$

In the second variation of the theme of Freud, we require linearity, but drop the requirement of interpolation. Although the operators need to be constructed using the data, we may want the approximation operator in some applications *not* interpolating the data; for example, when the data is noisy. Such operators, known as quasi-interpolatory operators, have been studied in the context of spline approximation by many mathematicians [3,4,6,21]. In the context of polynomial approximation, an obvious way to construct a quasi-interpolatory operator is to discretize a continuous, kernel-based operator. Let W be a Freud weight, $\{p_n\}$ be the system of orthonormal polynomials on $\mathbb{R}$ with respect to W^2, and for integer $n \geqslant 1$, let

$$V_n(x,t) := \sum_{k=0}^{n-1} p_k(x)p_k(t) + \sum_{k=n}^{2n-1} \left(2 - \frac{|k|}{n}\right) p_k(x)p_k(t). \tag{5.6}$$

The shifted average operator v_n is defined by

$$v_n(f,x) := \int_{\mathbb{R}} V_n(x,t) f(t) W^2(t)\, dt. \tag{5.7}$$

Freud proved that the operators v_n are uniformly bounded in all weighted L^p norms, and used this fact extensively in his study of weighted polynomial approximation (cf. [50]). In [57], we proved that the operators $\tau_{n,m}$ obtained by discretizing the integral in (5.7) using the Gauss–Jacobi quadrature formula based on zeros of p_m, $(2+\delta)n \leqslant m \leqslant Ln$ for some $\delta, L > 0$, are uniformly bounded in the sense that for every f

with $Wf \in C_0(\mathbb{R})$,

$$||W\tau_{n,m}(f)||_{\infty,\mathbb{R}} \leqslant c||Wf||_{\infty,\mathbb{R}}. \tag{5.8}$$

This work is further generalized to the context of different weights in [40,41,46]. It was observed in [56] that a Marcinkiewicz–Zygmund inequality, together with the uniform boundedness of kernel operators similar to those in (5.7) lead to the uniform boundedness of the discretized operators. In [52], we proved the necessary Marcinkiewicz–Zygmund inequality in the case of an arbitrary, sufficiently dense point set on the real line in the case when the weight function is $\exp(-|x|^\alpha)$, $\alpha > 1$. Therefore, for these weights, it is possible to discretize the operators in (5.7) using quadrature formulas based on an arbitrary system of points on $\mathbb{R}$. A generalization to other weights is in progress [54].

6. Multivariate approximation

The very first results on multivariate polynomial approximation were probably obtained by Dzrbasyan and Tavadyan [18]. They required a stringent requirement on the target function g, namely, $W^{-1}g \in C_0(\mathbb{R})$. A search of Web of Science showed only one other paper on multivariate weighted approximation [34] on a simplex. Multivariate weighted polynomial approximation appears to be a very fruitful area for future research. In [50], we have illustrated one application to the theory of Gaussian networks.

In [51], we have given a more complete analogue of the results of Dzrbasyan and Tavadyan without the restrictive condition, valid for all Freud weights. We observe that in the statement of the Bernstein approximation problem, one approximates a function $g \in C_0(\mathbb{R})$ by weighted polynomials. Freud found it convenient to write $g = wf$, and interpret the approximation as polynomial approximation of f in a weighted norm. While such moduli of smoothness as defined by (2.5) utilize the differences of the function f, Freud's original attempts [25] utilized the differences of g. This turned out to be more useful for an amusing result in [51]. Let $s \geqslant 1$ be an integer, $g \in L^p(\mathbf{R}^s)$ for some p, $1 \leqslant p < \infty$ or be a continuous function on $\mathbf{R}^s$, vanishing at infinity. Let $m \geqslant 2$ be an even integer, $\mathbf{a} = (a_1, \dots, a_s) \in (0, \infty)^s$ and

$$E_{p,n,s}(m, \mathbf{a}; g) := \min \left\| g - \exp\left(- \sum_{k=1}^{s} a_k x_k^m \right) P \right\|_{p,\mathbb{R}^s},$$

where the minimum is taken over all polynomials P of s variables having coordinatewise degree at most n. We have proved that if $E_{p,n,s}(m, \mathbf{a}; g) = \mathcal{O}(n^{-\beta})$ for some $\beta > 0$ and $\mathbf{a} \in (0, \infty)^s$, then the same estimate holds also for *all* $\mathbf{a} \in (0, \infty)^s$. This result was used heavily [53] in the proof of a converse theorem for approximation by Gaussian networks.

7. Questions and conjectures

1. An interesting and surprising aspect of weighted polynomial approximation with respect to the weights $\exp(-|x|^\alpha)$, $\alpha > 1$ is that there are exact expressions for the order and type of entire functions in terms of the degree of approximation to their restrictions to $\mathbb{R}$ in the weighted norm (cf. [50, Chapter 7]). Nevertheless, approximation of analytic and entire functions remains relatively unexplored with general weight functions. In particular, it will be interesting to characterize the functions f for which

$$\limsup_{n\to\infty} E_{n,p}(W,f)^{1/n} < 1, \tag{7.1}$$

where, for a weight function W and $1 \leqslant p \leqslant \infty$,

$$E_{n,p}(W,f) \coloneqq \min_{P\in\Pi_n} ||(f-P)W||_{p,\mathbb{R}}.$$

One reason for this interest is that the zeros of polynomials of best approximation to functions *not* satisfying (7.1) exhibit a remarkable asymptotic behavior (cf. [1,60]).

2. It is well known [59,66] that under suitable conditions on $W(x) = \exp(-Q(x))$, there exists, for every integer $n \geqslant 1$, a unique probability measure $\mu_{W,n}$, supported on $[-1,1]$ that maximizes

$$\int\int \log|W(a_n x)W(a_n t)(x-t)|\, dv(x)\, dv(t)$$

among all compactly supported probability measures v supported on $\mathbb{R}$, where a_n is defined by (2.3). It will be interesting to prove the following analogue of Theorem 5.1.

Let $x_{k,n}$ be distinct points on $\mathbb{R}$, W be a weight function such that the measures $\mu_{W,n}$ are supported on $[-1,1]$. The following are equivalent.

(a) *To every f with $Wf \in C_0(\mathbb{R})$ and $\varepsilon > 0$, there exists a sequence of polynomials $r_n \in \Pi_{n(1+\varepsilon)}$ such that $r_n(x_{k,n}) = f(x_{k,n})$ for $k = 1, \dots, n$, and $||(f-r_n)W||_{\infty,\mathbb{R}} \to 0$ as $n \to \infty$.*

(b) *We have*

$$\limsup_{n\to\infty} \frac{\#\{k\colon\ x_{k,n}/a_n \in I_n\}}{n\mu_{W,n}(I_n)} \leqslant 1 \tag{7.2}$$

for every sequence of intervals $I_n \subseteq [-1,1]$ for which $\lim_{n\to\infty} n\mu_{W,n}(I_n) \to \infty$, and

$$\liminf_{n\to\infty} n\mu_{W,n}([x_{k+1,n}/a_n, x_{k,n}/a_n]) > 0, \quad 1 \leqslant k \leqslant n. \tag{7.3}$$

It is worth mentioning here that the location and distribution of node systems $\{x_{k,n}\}$ that provide a "good" interpolation process has been studied by Szabados [72] and Damelin [9].

3. It will be interesting to study multivariate weighted approximation with non-tensor product weights, for example, radial weights.

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